Introduction to Generalized p-value

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February 12, 2014

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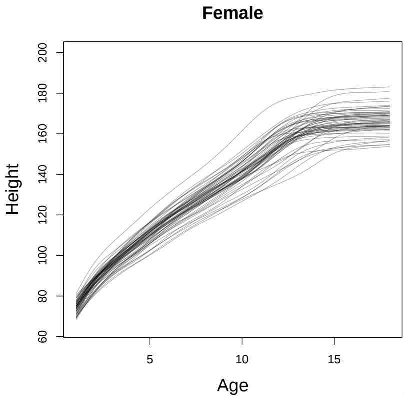
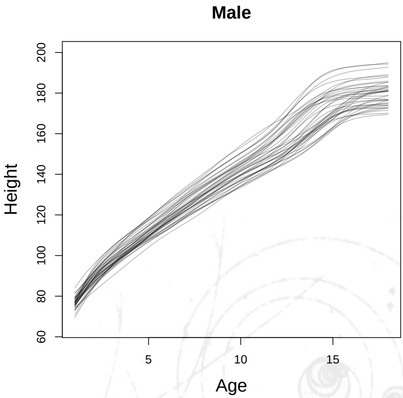
Comparing Two Groups of Curves

- Is there any difference between the two sets of curves?





Hypothesis Testing

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- How can we measure the confidence of our conclusion?

Hypothesis Testing

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- A typical statistical inference problem – Hypothesis Testing
- p-value is a widely used

Challenges

- Not easy to find a suitable test statistic
- Need to study the sampling distribution of the statistics
- Often involves many unknown parameters
- Consequence: Exact solutions only to limited number of problems

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- Reason: Many restrictions on the test statistic
- Generalized p-value relaxes some of the limitations

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Two Sample Mean Test

- The example I mentioned the beginning is similar to a two-sample t test
- We've collected two samples
  - $X_1, X_2, \ldots, X_m \stackrel{iid}{\sim} N(\mu_1, \sigma_1^2)$
  - $Y_1, Y_2, \ldots, Y_n \stackrel{iid}{\sim} N(\mu_2, \sigma_2^2)$, independent of X_i
- Test problem

$$H_0 : \theta = \mu_1 - \mu_2 = 0 \leftrightarrow H_a : \theta \neq 0$$

Two Sample Mean Test

- If we assume $\sigma_1^2 = \sigma_2^2$, it reduces to a problem we are familiar with

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}}, \ S_p^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}$$

- Under $H_0, T \sim t(m+n-2)$, so we reject null hypothesis if

$$|T| > t_{1-\alpha/2}(m+n-2)$$

- Equivalently, we calculate the p-value as

$$p = P(|T(X,Y)| > |T(x,y)|)$$

- Now, what if we drop the assumption of equal variance?

Behrens-Fisher Problem

- This is often referred to as the Behrens-Fisher Problem
- In this case the distribution of T depends on the "nuisance" parameters σ_1^2 and σ_2^2
- As a result, we cannot calculate the p-value $P(|T(X,Y)| > |T(x,y)|)$ exactly
- Some approximate solutions
  - Welch's approximation t
  - Likelihood ratio test
  - See Das Gupta (2008) and Wikipedia `http://en.wikipedia.org/wiki/Behrens-Fisher_problem`

Review of Test Statistic

- In hypothesis testing, we need to construct a test statistic T which satisfies the following properties
  - $T(\cdot)$ is a function only of sample X, so that T doesn't depend on unknown parameters
  - As a result, when plugging observed value of X into $T(\cdot)$, $T(x)$ is a constant (i.e., non-random)
  - Under H_0, the distribution of T is free of unknown parameters
  - $P(T > t|\theta)$ is nondecreasing in θ
- If satisfied, then the p-value $p = P(T(X) > T(x))$ can be calculated exactly

Generalized Test Variable

- We relax the first requirement, so that T can be a function of the random sample X, observed value x, and nuisance parameter η, written as $T(X; x, \eta)$. We now call $T(X; x, \eta)$ a generalized test variable.
- The other three requirements still hold
  - $T(x; x, \eta)$ is a constant, non-random and free of $\xi = (\theta, \eta)$
  - Under H_0, the distribution of T doesn't rely on ξ either
  - $P(T > t|\theta)$ is nondecreasing in θ for fixed x and η
- If we've found such a $T(X; x, \eta)$, then the generalized p-value can be calculated as

$$p = P(T(X; x, \eta) > T(x; x, \eta))$$

which actually doesn't depend on the nuisance parameter η

Solution to B-F Problem by G-p-value

- Consider the following generalized test variable with $\eta = (\sigma_1^2, \sigma_2^2)$ (Tsui and Weerahandi, 1989)

$$T(X, Y; x, y, \eta) = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \cdot \sqrt{\frac{\sigma_1^2 s_1^2}{m S_1^2} + \frac{\sigma_2^2 s_2^2}{m S_2^2}}$$

- We can verify

1. $T(x, y; x, y, \eta) = \bar{x} - \bar{y}$
2. Under $H_0 : \mu_1 = \mu_2$, the distribution of $T(X, Y; x, y, \eta)$ doesn't depend on $\theta = \mu_1 - \mu_2$ or $\eta = (\sigma_1^2, \sigma_2^2)$ (next slide)

Solution to B-F Problem by G-p-value

- We know the following facts

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right)$$

$$\frac{(m-1)S_1^2}{\sigma_1^2} \sim \chi^2_{m-1}, \ \frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}$$

- In the expression of T, given $\mu_1 = \mu_2$,

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0,1), \ \frac{\sigma_1^2 s_1^2}{m S_1^2} \sim \frac{m-1}{m} \frac{s_1^2}{\chi^2_{m-1}}, \ \frac{\sigma_2^2 s_2^2}{n S_2^2} \sim \frac{n-1}{n} \frac{s_1^2}{\chi^2_{n-1}}$$

Computation

- To compute the generalized p-value, simply simulate random numbers of T to approximate the probability

$$p = P(|T| > |\bar{x} - \bar{y}|)$$

- There is also an efficient method using numerical integration, see Tsui and Weerahandi (1989)

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Multivariate B-F Problems

- Now we move to multivariate case
- $X_1, X_2, \dots, X_m \overset{iid}{\sim} N_d(\mu_1, \Sigma_1)$
- $Y_1, Y_2, \dots, Y_n \overset{iid}{\sim} N_d(\mu_2, \Sigma_2)$, independent of X_i
- Test problem

$$H_0 : \theta = \mu_1 - \mu_2 = 0 \leftrightarrow H_a : \theta \neq 0$$

Solution by G-p-value

- More tricky and complicated
- The key idea is to generalize the following univariate test variable to multivariate scenario

$$T^2(X, Y; x, y, \eta) = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \left(\frac{\sigma_1^2 s_1^2}{m S_1^2} + \frac{\sigma_2^2 s_2^2}{m S_2^2} \right) \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$$

- Gamage et al. (2004) provides a solution

Some Notations

- $\bar{X}, \bar{Y}$ as the sample mean vector
- $W_1 = \sum_{i=1}^{m}(X_i - \bar{X})(X_i - \bar{X})' = (m-1)S_1, W_2 = (n-1)S_2$, the unscaled sample covariance matrices
- It can be derived that

$$\bar{X} - \bar{Y} \sim N_d\left(\mu_1 - \mu_2, \Sigma_p\right), \ \Sigma_p = \frac{1}{m}\Sigma_1 + \frac{1}{n}\Sigma_2$$

$$W_1 \sim W(\Sigma_1, m-1), \quad W_2 \sim W(\Sigma_2, n-1)$$

- $W(\Sigma, n)$ is the Wishart distribution, the generalization of χ^2 distribution to multivariate case

Solution by G-p-value

- Let

$$R_1 = (w_1^{-\frac{1}{2}}\Sigma_1 w_1^{-\frac{1}{2}})^{-\frac{1}{2}}(w_1^{-\frac{1}{2}}W_1 w_1^{-\frac{1}{2}})(w_1^{-\frac{1}{2}}\Sigma_1 w_1^{-\frac{1}{2}})^{-\frac{1}{2}}$$

R_2 alike, $Z_0 = \Sigma_p^{-\frac{1}{2}}(\bar{X} - \bar{Y})$, $\eta = (\Sigma_1, \Sigma_2)$, then the test variable is (Gamage et al., 2004)

$$T(X, Y; x, y, \eta) = Z_0' \left(\frac{1}{m} w_1^{\frac{1}{2}} R_1^{-1} w_1^{\frac{1}{2}} + \frac{1}{n} w_2^{\frac{1}{2}} R_2^{-1} w_2^{\frac{1}{2}}\right) Z_0$$

1. $T(x, y; x, y, \eta) = (\bar{x} - \bar{y})'(\bar{x} - \bar{y})$
2. Under H_0,
   $R_1 \sim W(I_d, m-1), R_2 \sim W(I_d, n-1), Z_0 \sim N_d(0, I_d)$, so the distribution of T is parameter-free

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To be continued...

I'll cover this topic in the final course presentation

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A Summary of G-p-value

- Provide a p-value in the presence of nuisance parameters
- Relax restrictions on test statistic
- Allow test variable depend on observed values and nuisance parameters, while the final p-value turns out to be parameter-free

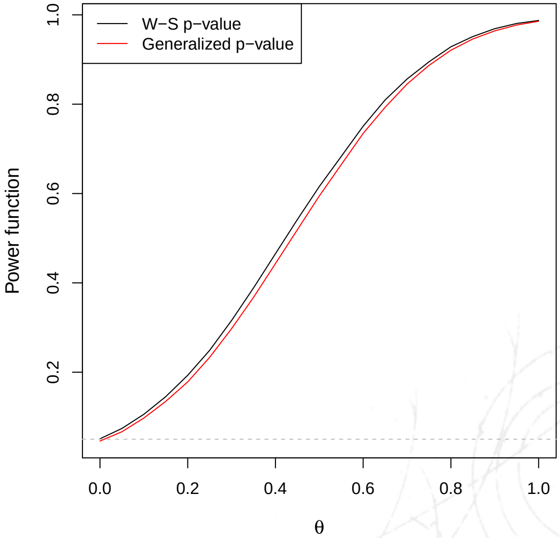
The Good Part

- Provides solutions to the Behrens-Fisher problem, both univariate case and multivariate case
- Also applicable to other problems where traditional method fails

The Not-So-Good Part

- Test variable depends on both random sample and observed value, sometimes hard to explain (in some sense related to Fiducial Inference)
- Tends to give conservative result (slightly less powerful)
- Distribution of test variable usually has no closed form (but easy for sampling)

The Not-So-Good Part



References

- Tsui, K. W., & Weerahandi, S. (1989). Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters. Journal of the American Statistical Association, 84(406), 602-607.
- Gamage, J., Mathew, T., & Weerahandi, S. (2004). Generalized p-values and generalized confidence regions for the multivariate Behrens–Fisher problem and MANOVA. Journal of Multivariate Analysis, 88(1), 177-189.
- DasGupta, A. (2008). Asymptotic theory of statistics and probability. Springer.

Thank you!